

# Parameterization of temperature and spectral distortions in future CMB experiments

Cyril Pitrou<sup>1</sup> and Albert Stebbins<sup>2</sup>

<sup>1</sup>*Institut d'Astrophysique de Paris, CNRS UMR 7095,  
UPMC Univ Paris 06, 98 bis Bd Arago, 75014 Paris, France*

*Sorbonne Universités, Institut Lagrange de Paris, 98 bis Bd Arago, 75014 Paris, France*

<sup>2</sup>*Theoretical Astrophysics Group, Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510*

(Dated: January 6, 2015)

CMB spectral distortions are induced by Compton collisions with electrons. We review the various schemes to characterize the anisotropic CMB with a non-Planckian spectrum. We advocate using logarithmically averaged temperature moments as the preferred language to describe these spectral distortions, both for theoretical modeling and observations. Numerical modeling is simpler, the moments are frame-independent, and in terms of scattering the mode truncation is exact.

PACS numbers: 95.30.Jx, 98.70.Vc, 98.80.-k, 98.80.Es

The Planck Surveyor's measurements of the cosmic microwave background (CMB) have opened a new era in the analysis and extraction of cosmological information from CMB data. The CMB spectrum, *i.e.* departures from a black-body radiation (BB), have not only been used to detect “point” sources from the SZ effect [1], but we now have angular maps of SZ and other foregrounds to work with [2]. We are entering an era where the CMB itself, both the intensity and polarization patterns, and not even including foregrounds, should no longer be treated as two-dimensional (angle dependence) but rather three dimensional including the spectral (frequency) dependence. Cosmological information from all three dimensions needs to be extracted optimally [3] and theoretical modeling of this third dimension must be done accurately.

The frequency dependent brightness temperature is a complete description of the CMB which evolves according to the Boltzmann equation, and is a frame dependent quantity. When departures from BB are measurable, then a local definition of the “temperature anisotropy”,  $\delta T$ , becomes ambiguous. Ideally one would like to decompose into a truncated series of modes, reducing partial into simpler ordinary differential equations. Here we review the different proposed decompositions and argue that the logarithmically averaged temperature moments (LAM) first used in Ref. [4] and further developed in Ref. [5], is the best decomposition and should be used for future theoretical modeling. These LAM can be added to the spectral templates used to fit the CMB plus foregrounds.

For the two angular dimensions, the assumed statistical isotropy guarantees that the spectrum,  $C_\ell$ , the bispectrum,  $B_{\ell,\ell',\ell''}$ ,  $\dots$  are sufficient two-point, three-point,  $\dots$  statistics. These spectra are based on a spherical harmonic decomposition of the angular dependence. For the spectral (frequency) decomposition we argue that the LAM are the most appropriate compact representation of the CMB spectral distortions because *i)* it leads to a frame invariant description of spectral distortions, which is independent from our local velocity, and *ii)* it is

the variable with which the non-linear numerical integration is the simplest and was thus recently made possible for several groups [6, 7].

## General formalism for the description of a spectrum

*Temperature transform.* The distribution function of radiation is a function of the position in space-time, the direction of propagation and the energy  $E$  of radiation. It is of the form  $n(E, \dots)$  (here  $\dots$  indicates all the non-spectral dependence and is often omitted below). In previous literature [8–11] the starting point for the description of the spectral dependence is to consider that the CMB spectrum is a superposition of BBR with different temperatures, given by the distribution  $p(T, \dots)$ , such that

$$n(E, \dots) = \int_0^\infty dT p(T, \dots) \mathcal{N}\left(\frac{E}{T}\right) \quad (1)$$

with  $\mathcal{N}(x) \equiv 1/(\exp(x) - 1)$ . If  $\int_0^\infty p(T) dT \neq 1$  the distribution is said to be “gray”. Ref. [5] gives a full treatment of grayness and there it is shown that an initially non-gray distribution with only Compton-type interactions will remain non-gray. Henceforth we consider only non-gray distributions (see Ref. [12] for an example of a process inducing grayness). One can characterize the shape of the spectrum by the moments of the distribution  $p(T)$ . One thus defines

$$\bar{T}_{(p)} \equiv \left( \int_0^\infty T^p p(T) dT \right)^{\frac{1}{p}}. \quad (2)$$

Different authors have concentrated on following only specific moments. The most commonly used are the Rayleigh-Jeans temperature,  $\bar{T}_{RJ} \equiv \bar{T}_{(1)}$  [11]; the number density temperature,  $\bar{T}_n \equiv \bar{T}_{(3)}$  [13–15]; and the bolometric temperature  $\bar{T}_b \equiv \bar{T}_{(4)}$  [6, 16, 17]; giving respectively the low frequency brightness, the number density of photons, and the energy density in photons. Indeed, using

Eq. (1) we find  $T_{(p)}^p \propto \int_0^\infty n(E) E^{p-1} dE$  for  $p \geq 2$ . Note that if the distribution function has a chemical potential, as in the case of a general Bose-Einstein distribution, the low energy limit is then a constant  $([\exp(\mu/T) - 1]^{-1})$ , and it is thus impossible to describe such distribution as a superposition of BBR like in Eq. (1) whose low energy limit is  $\propto \bar{T}_{RJ}/E$ . See however the appendix for the treatment of the case where the effect of the chemical potential is negligible at low energy.

However in Ref. [5] an alternative description of the spectral distortions was proposed. At the basis of the formalism, is the use of the variable  $\mathcal{T} \equiv \ln T$  (where a reference unit of temperature is implicit) whose distribution is  $q(\mathcal{T}) \equiv Tp(T)$ . The logarithmically averaged temperature is then simply defined by

$$\bar{\mathcal{T}} \equiv \langle \mathcal{T} \rangle \equiv \ln \bar{T}, \quad \text{with} \quad \langle f \rangle \equiv \int_{-\infty}^{\infty} d\mathcal{T} f(\mathcal{T}) q(\mathcal{T}). \quad (3)$$

*Spectral moments.* The spectral distortions are characterized by the moments (LAM) of  $q(\mathcal{T})$ : the moments about 0,  $\{\eta_p\}$ ; the central moments,  $\{u_p\}$ ; and the moments about a reference temperature,  $\{d_p\}$ , *i.e.*

$$\eta_p \equiv \langle \mathcal{T}^p \rangle, \quad u_p \equiv \langle (\mathcal{T} - \bar{\mathcal{T}})^p \rangle, \quad d_p \equiv \langle (\mathcal{T} - \mathcal{T}_0)^p \rangle \quad (4)$$

where  $\mathcal{T}_0 \equiv \ln T_0$  and  $T_0$  is an arbitrary reference temperature, usually chosen close to the mean. By construction,  $u_1 = 0$ , and since the spectrum is non-gray  $\eta_0 = d_0 = u_0 = 1$ . Using  $\mathcal{T} = (\mathcal{T} - \bar{\mathcal{T}}) + \bar{\mathcal{T}}$  and  $\mathcal{T} = (\mathcal{T} - \mathcal{T}_0) + \mathcal{T}_0$ , the moments of eq. (4) are related by Leibniz-type relations

$$u_p = \mathcal{B}_p[-\bar{\mathcal{T}}, \{\eta_k\}] = \mathcal{B}_p[-d_1, \{d_k\}] \quad (5)$$

$$\eta_p = \mathcal{B}_p[\bar{\mathcal{T}}, \{u_k\}] \quad d_p = \mathcal{B}_p[d_1, \{u_k\}] \quad (6)$$

where  $\mathcal{B}_p[A, \{B_k\}] \equiv \sum_{m=0}^p \binom{p}{m} A^{p-m} B_m$ . The meaning of the moments is clear as one can reconstruct the spectrum by

$$n(E) = \sum_{m=0}^{\infty} \frac{d_m}{m!} D^m \mathcal{N}\left(\frac{E}{T_0}\right) = \sum_{m=0}^{\infty} \frac{u_m}{m!} D^m \mathcal{N}\left(\frac{E}{\bar{T}}\right) \quad (7)$$

where  $D^m \mathcal{N}(x) \equiv (-1)^m d^m \mathcal{N}(x)/d \ln(x)^m$ . Thus  $\{d_m\}$  and  $\{u_m\}$  are the coefficients of a generalized Fokker-Planck expansion around  $T_0$  and  $\bar{T}$ , respectively. The  $u_p$  are frame independent, but this is not the case for the other types of moments [5]. While this is an infinite expansion the lower order moments do not depend dynamically on the higher order moments (see below). The observed spectrum as a function of frequency and direction requires knowledge of the observer frame because of the Doppler effect and associated aberration, so one must also know  $\bar{\mathcal{T}}$  and thus  $d_1 = \bar{\mathcal{T}} - \mathcal{T}_0$ . Truncation is useful because it is the lowest order moments which are the most evident: the 1st moment give the “temperature perturbation”  $\delta T/T = \delta \ln T = d_1$ ; while the 2nd moment

give the Compton  $y$  distortion,  $y = \frac{1}{2} u_2 = \frac{1}{2} (d_2 - d_1^2)$ . These two moments are the ones most relevant for current observations.

*Spectral moment of a polarized spectrum.* Linear polarization will be generated by Compton scattering and the previous formalism can be extended to describe the polarization spectrum [5]. We use a  $2 \times 2$  matrix-valued distribution function,  $n^{ab}(E)$ , to describe both intensity and polarization (see e.g. Ref. [18] for a review). The trace ( $n$ ), the symmetric traceless part ( $n^{(ab)}$ ), and the antisymmetric part give, respectively, the intensity, the linear polarization, and the circular polarization. The latter is not generated by Compton collisions so we set it to zero. We thus have  $n^{ab}(E) = \delta^{ab} n(E) + n^{(ab)}(E)$ . A matrix-valued distribution of BBR  $q^{ab}(\mathcal{T})$  is defined by

$$n^{ab}(E) = \int_{-\infty}^{\infty} d\mathcal{T} q^{ab}(\mathcal{T}) \mathcal{N}(E e^{-\mathcal{T}}) \quad (8)$$

and its matrix-valued moments  $\{d_p^{ab}\}, \{\eta_p^{ab}\}, \{u_p^{ab}\}$  can be generalized from the  $\{d_p\}, \{\eta_p\}, \{u_p\}$ , for which a trace and a symmetric traceless part can be defined. The relations (5,6) are then straightforwardly extended for linear polarization.

From the structure of the Compton collision term, it can be shown [5] that  $d_0^{(ab)} = \eta_0^{(ab)} = 0$  if initially so, but  $u_1^{(ab)} \neq 0$ . The set of variables for the polarized part is thus simply the set of  $\{u_p^{(ab)}\}_{p \geq 1}$  as they are frame independent. Compared to the intensity, the main difference is that there is no temperature to be defined for polarization, but there is the non-vanishing moment  $u_1^{(ab)}$  which is the dominant one. A common misstatement or misunderstanding consists in treating this moment as a temperature perturbation, and to use the definition  $\Theta^{(ab)} \equiv u_1^{(ab)}$ , but strictly speaking, it is a pure spectral distortion, and as such frame independent. In Ref. [14], it is called the “temperature part” of the polarization, as opposed to the primary spectral distortion  $u_2^{(ab)} = d_2^{(ab)} - 2d_1 d_1^{(ab)}$ .

*Extraction of moments from a measured spectrum.* In appendix A of Ref. [5], it is shown that, from a spectrum  $n(E)$ , the moments  $\{\eta_p\}$  can be obtained from

$$\eta_p = \sum_{m=0}^p \binom{p}{m} \tilde{\kappa}_{p-m} I_p[n(E)] \quad (9)$$

$$I_p[n(E)] = \int_{-\infty}^{\infty} (\ln E)^p [D^2 n(E) - D n(E)] d \ln E$$

and the central moments are deduced from (5). The lowest order  $\tilde{\kappa}_p$  are computed in [5]. If the signal is sampled in several bands, and if the number of bands is large enough (as should be the case for future CMB experiments [19]), then linear combinations of these signals with appropriate weights would be equivalent to the numerical integrations  $I_p[n^{ab}(E)]$ , and would thus allow

determination of the temperature and the first spectral moments. This method is straightforwardly extended for the extraction of polarization moments  $\{\eta_p^{(ab)}\}$  and then the  $\{u_p^{(ab)}\}$ .

*Discussion on the choice of a set of variables.* It is clear that since the  $\{u_p\}_{p \geq 2}$  are frame invariant they are good candidates to describe the spectral distortions. The use of  $d_1$  for the temperature perturbation is then natural as it fits into this formalism. However, one might wonder if this is the only set of variables with such appealing properties. Starting from the moments defined in (2), we can relate these to the  $\{d_p\}$  and  $\{u_p\}$  by

$$\langle T^p \rangle = (\bar{T}_{(p)})^p = T_0^p \sum_m \frac{p^m d_m}{m!} = \bar{T}^p \sum_m \frac{p^m u_m}{m!}. \quad (10)$$

It appears clearly that, for a given  $p$ , the temperature  $\bar{T}_{(p)}$  can be used to define a temperature perturbation and the moments

$$\Theta_{(p)} \equiv \bar{T}_{(p)}/T_0 - 1, \quad M_{(p),m} \equiv \langle (T - \bar{T}_{(p)})^m \rangle \bar{T}_{(p)}^{-m}. \quad (11)$$

The  $\{M_{(p),m}\}_{m \geq 2}$  would be as good as the  $\{u_m\}_{m \geq 2}$  to describe the spectral distortions, since they are obviously frame invariant as they involve only an (infinite) sum of products of the  $\{u_p\}$ . In the next section, we argue that to decide which set of variables should be used, one should examine the dynamical evolution, and choose the one which has the simplest structure, and for which numerical integration is simplified.

### Dynamical evolution of spectral moments

*General form of the Boltzmann equation.* The general form of the Boltzmann equation is (again we omit the dependence in  $(E, \dots)$  for brevity)

$$L^{ab}[n] \equiv \frac{\mathcal{D}n^{ab}}{\mathcal{D}\eta} + \frac{d \ln E}{d\eta} \frac{\partial n^{ab}}{\partial \ln E} = C^{ab}[n] \quad (12)$$

where the convective derivative  $\mathcal{D}/\mathcal{D}\eta$  acts on all the dependence except the spectral dependence, and accounts for the effect of free streaming. The collision term can also be described by its moments  $\{\eta_p^{C,ab}\}, \{u_p^{C,ab}\}, \{d_p^{C,ab}\}$  which are related by relations similar to (5) and (6). In order to find the evolution of the  $\{u_p^{ab}\}$ , it proves simpler to first derive from (12) the evolution of the  $\{d_p^{ab}\}$ , and we get

$$\frac{\mathcal{D}d_m^{ab}}{\mathcal{D}\eta} = m d_{m-1}^{ab} \frac{d \ln E}{d\eta} + d_m^{C,ab}. \quad (13)$$

So for the temperature perturbation, the trace of  $m = 1$ :

$$\frac{\mathcal{D}d_1}{\mathcal{D}\eta} = \frac{d \ln E}{d\eta} + d_1^C. \quad (14)$$

If the spectrum is initially non-gray, and radiation is only subject to Compton scattering, it remains so and this property translates to  $d_0^{C,ab} = 0$ . The moments  $\{d_p^{C,ab}\}$  can be read off the collision term (see e.g. Ref. [20]), and as long as the thermal effects are ignored (or treated separately from the Kompaneets equation [21]), they are linear in the variables  $\{d_p\}$  which describe the radiation spectrum. However, they still couple non-linearly to the baryons bulk velocity [18, 22].

From the relations (5), one infers that

$$\begin{aligned} \frac{\mathcal{D}u_p^{ab}}{\mathcal{D}\eta} &= \sum_{m=1}^p \binom{m}{p} (-d_1)^{p-m} [d_m^{C,ab} - m d_{m-1}^{ab} d_1^C] \\ &= u_p^{C,ab} - p u_{p-1}^{ab} d_1^C. \end{aligned} \quad (15)$$

This system of equation is closed at any order  $p$ , since the equation-of-motion for  $u_p^{ab}$  depends only on  $u_{p'}^{ab}$  for  $p' \leq p$ . One can truncate this system of equations at any order with no approximation!

*Doppler, SZ effect and y-type distortion.* At 1st order one needs only the temperature perturbation  $d_1$  and  $u_1^{(ab)} = d_1^{(ab)}$ . At 2nd order, one adds the spectral distortions  $u_2$  and  $u_2^{(ab)}$ , and this distortion, known in this context as the non-linear kinetic SZ effect [13, 15], is generated by the r.h.s. of (15) with  $p = 2$ .

The distortion generated by the thermal SZ effect [23] is also captured by  $u_2$  and the usual  $y$  parameter associated with it is related by  $y \equiv \frac{1}{2}u_2 = \frac{1}{2}(d_2 - d_1^2)$ . A polarized  $y$ -type distortion can also be defined [14, 15, 24] and is related to the moments by  $y^{(ab)} \equiv \frac{1}{2}u_2^{(ab)} = \frac{1}{2}(d_2^{(ab)} - 2d_1 d_1^{(ab)})$ .

*Structure of the numerics.* Eq. (15) shows that

1. spectral distortions are affected only by the collision term, as they remain unaffected by metric perturbations (see also Refs. [5, 13, 14]);
2. metric perturbations, which enter through the red-shifting term  $d \ln E / d\eta$  affect only the evolution of the temperature perturbation  $d_1$ , and more importantly do not couple non-linearly with  $d_1$  [Eq. (14)];
3. the collision term for the evolution of  $u_p^{ab}$  [the r.h.s of (15)], contains only terms of the form  $d_1^{p-k} u_k^{ab}$  with  $k \leq p$  (see Ref. [5] for more details) multiplied by powers of the baryons bulk velocity. Therefore it restricts the non-linearities to products of at most  $p$  factors of spectral moments, when considering the evolution of the moment of order  $p$ . N.B. for  $p = 1$  the collision term ( $d_1^{C,ab}$ ) is linear in the moments.

Any other parameterization of the distortion based on the  $M_{(p),n}$  defined in (11) would conserve property (1). However, property (3) would be lost with the  $M_{(p),n}$ . The loss of this property is, in principle, not a serious problem for the numerical integration, since interactions are

localized in time by the visibility function. However, this would lead to unnecessary complications when going to higher orders of perturbations and thus higher moments. Our first argument here is that the simplest is the best.

Our second argument is that property (2) is crucial for the numerical integration since redshifting effects are not localized in time. Indeed, by avoiding a non-linear coupling between the temperature perturbations and the metric perturbations, the numerical integration is made possible even at the non-linear level as it avoids coupling between the angular moments of the temperature perturbations with the metric perturbation [6]. Finding a form of the Boltzmann equation that satisfies this property, was the key to a successful numerical integration at second order [6, 7]. With the present formalism, this property arises naturally for the variable  $d_1$ . Metric perturbations would also affect the geodesic and lead to time-delay and lensing effects, but these can be treated separately [25, 26]. There would be of course other variables for which property (2) holds. For instance, defining  $\tilde{\Theta}_{(p)} \equiv \ln(1 + \Theta_{(p)})$ , one obtains from (10) that the variables

$$\tilde{\Theta}_{(p)} = d_1 + \frac{1}{p} \ln \left( 1 + \sum_{m \geq 2} \frac{p^m u_m}{m!} \right) \quad (16)$$

obviously satisfy property (2) but not property (3). Up to second order in cosmological perturbations (neglecting  $\{u_p\}_{p \geq 3}$ ) the definitions for the most common temperatures are related by  $d_1 \simeq \tilde{\Theta}_n - \frac{3}{2}u_2 \simeq \tilde{\Theta}_b - 2u_2 \simeq \tilde{\Theta}_{RJ} - \frac{1}{2}u_2$ . As an illustration, if we consider the bolometric temperature perturbation  $\Theta_b$ , then up to second order in cosmological perturbations, only  $d_1$  and  $u_2$  need to be kept, and one finds that  $\Theta_b \simeq d_1 + \frac{1}{2}d_1^2 + 2u_2$ , but  $\tilde{\Theta}_b \simeq d_1 + 2u_2$ . This motivated the use of  $\tilde{\Theta}_b$  instead of  $\Theta_b$  in the final output of Ref. [6], since property (2) is satisfied for the former and not for the latter. Similarly, for the fractional perturbation to the energy density, one finds up to second order in cosmological perturbations  $\Delta \simeq 4[d_1 + 2d_1^2 + 2u_2]$ , and using  $\tilde{\Delta} \equiv \ln(1 + \Delta)$ , we find  $\tilde{\Delta} \simeq 4(d_1 + 2u_2) = 4\tilde{\Theta}_b$ . Again this motivated the use of  $\tilde{\Delta}$  instead of  $\Delta$  in the intermediate numerics of Ref. [6], so as to keep property (2) satisfied. A final example can be made with the fractional energy density perturbation of linear polarization. One finds  $\Delta^{(ab)} \simeq 4[d_1^{(ab)}(1 + 4d_1) + 2u_2^{(ab)}]$ , and the non-linear term  $d_1^{(ab)}d_1$  will induce a non-linear coupling of the type  $d_1^{(ab)}d \ln E/d\eta$  in the evolution equation of  $\Delta^{(ab)}$ . However, using  $\tilde{\Delta}^{(ab)} \equiv \Delta^{(ab)}(1 - 4d_1)$ , this non-linear coupling disappears [27] and property (2) is recovered. In all these three examples, property (2) can be restored with an ad-hoc change of variable, but property (3) is not satisfied, due to the term in  $u_2$  for the first two examples, and due to the term  $u_2^{(ab)}$  for the last one. It implies in particular that the evolution equation for the lowest

order moment in this description, *i.e.* their temperature perturbation, has a collision term which is not linear in the moments of radiation.

*Conclusion.* The essential properties described above for the structure of dynamical equations are only met with the set of variables made of  $d_1$ ,  $\{u_p\}_{p \geq 2}$  and  $\{u_p^{(ab)}\}_{p \geq 1}$ . Furthermore, the moments which characterize the spectral distortions are frame independent and thus do not depend on our local velocity. Only the angular dependence is affected by the choice of frame due to aberration effects. We strongly recommend that these moments should be used to parameterize the CMB spectrum. Furthermore, the analysis of spectral distortions from thermal effects, and other processes [28] could be rephrased in this unified language.

*Acknowledgements.* C.P. thanks J. Chluba, C. Fidler, Z. Huang, S. Renaux-Petel, G. Pettinari, J.-P. Uzan, and F. Vernizzi for fruitful discussions. This work was supported by French state funds managed by the ANR within the Investissements d'Avenir programme under reference ANR-11-IDEX-0004-02. AS was supported by the DOE at Fermilab under Contract No. DE-AC02-07CH11359.

## Appendix

*Distribution functions with a chemical potential.* At redshifts higher than  $z \simeq 10^4$  and smaller than  $z \simeq 3 \times 10^5$ , the thermalization of photons with an excess of energy with respect to a Planck spectrum results in a Bose-Einstein distribution with a chemical potential [29]. However, at low energies, processes such as double Compton emission and Bremsstrahlung are enough to modify the spectrum and remove the effect of the chemical potential so that the low energy limit of the distribution is still  $\propto 1/E$ . It is indeed approximately described by a Bose-Einstein distribution with an energy dependent chemical potential. At high energies, this chemical potential converges to a constant, but at low energies it is suppressed, and the transition from one regime to the other is governed by a cut-off  $x_c = E_c/T$  [29]. The distribution function of this ansatz is approximately of the form

$$n(x) = \frac{1}{e^{x+\mu(x)} - 1}, \quad \mu(x) = \mu_\infty e^{-x_c/x}. \quad (17)$$

with  $x \equiv E/T$ . A typical cut-off is  $x_c = 0.01$  and one can check that when  $\mu_\infty \rightarrow 0$ , this distribution approaches a BBR as the various moments decrease, as expected. In Fig. 1 we plot the first moments as a function the chemical potential  $\mu_\infty$  to illustrate this convergence.

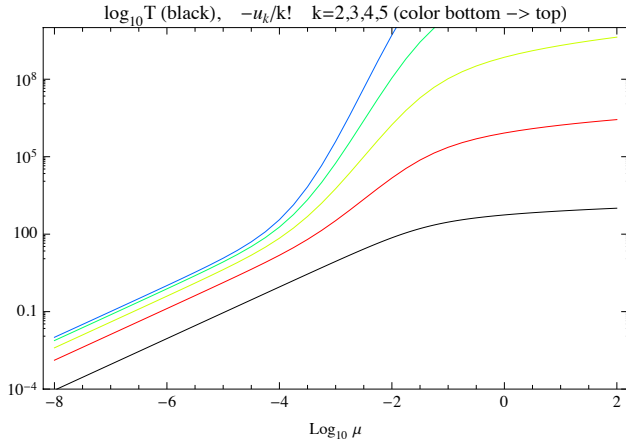


FIG. 1: The main moments for a pseudo Bose-Einstein distribution (17) with cut-off  $x_c = 0.01$ . In black :  $\log_{10} \bar{T}$ . In colors :  $-u_k/k!$  for  $k = 2, 3, 4, 5$  (from bottom to top).

- [2] P. Ade et al. (Planck Collaboration) (2013), 1303.5081.
- [3] J. Chluba and D. Jeong, MNRAS **438**, 2065 (2014), 1306.5751.
- [4] J. Goodman, Phys.Rev.D **52**, 1821 (1995).
- [5] A. Stebbins (2007), astro-ph/0703541.
- [6] Z. Huang and F. Vernizzi, Phys.Rev.Lett. **110**, 101303 (2013).
- [7] G. W. Pettinari, C. Fidler, R. Crittenden, K. Koyama, and D. Wands, JCAP **1304**, 003 (2013).
- [8] Y. B. Zel'dovich, A. F. Illarionov, and R. A. Sunyaev, Soviet JETP **35**, 643 (1972).
- [9] K. L. Chan and B. J. T. Jones, Ap.J. **198**, 245 (1975).
- [10] L. Salas, Ap.J. **385**, 288 (1992).
- [11] J. Chluba and R. Sunyaev, A.&A. **424**, 389 (2003).
- [12] G. Ellis, R. Poltis, J.-P. Uzan, and A. Weltman, Phys.Rev.D **87**, 103530 (2013).
- [13] C. Pitrou, F. Bernardeau, and J.-P. Uzan, JCAP **1007**, 019 (2010).
- [14] A. Naruko, C. Pitrou, K. Koyama, and M. Sasaki, Class.Quant.Grav. **30**, 165008 (2013).
- [15] S. Renaux-Petel, C. Fidler, C. Pitrou, and G. W. Pettinari, JCAP **1403**, 033 (2014), 1312.4448.
- [16] C. Pitrou, J.-P. Uzan, and F. Bernardeau, JCAP **1007**, 003 (2010).
- [17] P. Creminelli, C. Pitrou, and F. Vernizzi, JCAP **1111**, 025 (2011).
- [18] C. G. Tsagas, A. Challinor, and R. Maartens, Phys.Rept. **465**, 61 (2008).
- [19] P. Andre et al. (PRISM Collaboration) (2013).
- [20] M. Beneke and C. Fidler, Phys.Rev.D **82**, 063509 (2010).
- [21] A. Kompaneets, JETP **4**, 730 (1957).
- [22] C. Pitrou, Class.Quant.Grav **26**, 065006 (2009).
- [23] Y. B. Zeldovich and R. A. Sunyaev, Astroph. Sp. Sci. **4**, 301 (1969).
- [24] R. A. Sunyaev and I. B. Zeldovich, MNRAS **190**, 413 (1980).
- [25] W. Hu and A. Cooray, Phys.Rev.D **63**, 023504 (2001).
- [26] Z. Huang and F. Vernizzi, Phys.Rev. **D89**, 021302 (2014), 1311.6105.
- [27] C. Fidler, G. W. Pettinari, M. Beneke, R. Crittenden, K. Koyama, et al., JCAP **07**, 011 (2014), 1401.3296.
- [28] J. Chluba and R. Sunyaev, MNRAS **419**, 1294 (2012), 1109.6552.
- [29] J. Chluba, MNRAS **440**, 2544 (2014), 1312.6030.